

LOG MINIMAL MODELS ACCORDING TO SHOKUROV

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ABSTRACT. Following Shokurov's ideas, we give a short proof of the following klt version of his result: termination of terminal log flips in dimension d implies that any klt pair of dimension d has a log minimal model or a Mori fibre space. Thus, in particular, any klt pair of dimension 4 has a log minimal model or a Mori fibre space.

1. INTRODUCTION

All the varieties in this paper are assumed to be over an algebraically closed field k of characteristic zero. We refer the reader to section 2 for notation and terminology.

Shokurov [5] proved that the log minimal model program (LMMP) in dimension $d - 1$ and termination of terminal log flips in dimension d imply existence of a log minimal model or a Mori fibre space for any lc pair of dimension d . Following Shokurov's method and using results of [2], we prove that termination of terminal log flips in dimension d implies existence of a log minimal model or a Mori fibre space for any klt pair of dimension d .

In this paper, by termination of terminal log flips in dimension d we will mean termination of any sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips/ Z starting with a d -dimensional klt pair $(X/Z, B)$ which is terminal in codimension ≥ 2 .

Theorem 1.1. *Termination of terminal log flips in dimension d implies that any klt pair $(X/Z, B)$ of dimension d has a log minimal model or a Mori fibre space.*

Corollary 1.2. *Any klt pair $(X/Z, B)$ of dimension 4 has a log minimal model or a Mori fibre space.*

Note that, in the corollary, when $(X/Z, B)$ is effective (eg of nonnegative Kodaira dimension), log minimal models are constructed in [1] using different methods.

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2. BASICS

Let k be an algebraically closed field of characteristic zero. For an \mathbb{R} -divisor D on a variety X over k , we use D^\sim to denote the birational transform of D on a specified birational model of X .

Definition 2.1. A pair $(X/Z, B)$ consists of normal quasi-projective varieties X, Z over k , an \mathbb{R} -divisor B on X with coefficients in $[0, 1]$ such that $K_X + B$ is \mathbb{R} -Cartier, and a projective morphism $X \rightarrow Z$. $(X/Z, B)$ is called log smooth if X is smooth and $\text{Supp } B$ has simple normal crossing singularities.

For a prime divisor D on some birational model of X with a nonempty centre on X , $a(D, X, B)$ denotes the log discrepancy. $(X/Z, B)$ is terminal in codimension ≥ 2 if $a(D, X, B) > 1$ whenever D is exceptional/ X . Log flips preserve this condition but divisorial contractions may not.

Let $(X/Z, B)$ be a klt pair. By a log flip/ Z we mean the flip of a $K_X + B$ -negative extremal flipping contraction/ Z . A sequence of log flips/ Z starting with $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \rightarrow Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$ -flip/ Z and B_i is the birational transform of B_1 on X_1 , and $(X_1/Z, B_1) = (X/Z, B)$. By termination of terminal log flips in dimension d we mean termination of such a sequence in which $(X_1/Z, B_1)$ is a d -dimensional klt pair which is terminal in codimension ≥ 2 . Now assume that $G \geq 0$ is an \mathbb{R} -Cartier divisor on X . A sequence of G -flops/ Z starting with $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \rightarrow Z_i \leftarrow X_{i+1}$ is a G_i -flip/ Z such that $K_{X_i} + B_i \equiv 0/Z_i$ where G_i is the birational transform of G on $X = X_1$.

Remark 2.2. We borrow a result of Shokurov [5, Corollary 9, Addendum 4] concerning extremal rays. Let $(X/Z, B)$ be a \mathbb{Q} -factorial klt pair and F a reduced divisor on X . Then, there is $\epsilon > 0$ such that if $G \geq 0$ is an \mathbb{R} -divisor supported in F satisfying

- (1) $\|G\| < \epsilon$ where $\|\cdot\|$ denotes the maximum of coefficients, and
- (2) $(K_X + B + G) \cdot R < 0$ for an extremal ray R ,

then $(K_X + B) \cdot R \leq 0$. This follows from certain numerical properties of log divisors such as [5, Proposition 1] which is essentially the boundedness of the length of an extremal ray. Moreover, ϵ can be chosen such that for any \mathbb{R} -divisor $G' \geq 0$ supported in F and any sequence

$X_i \dashrightarrow X_{i+1}/Z_i$ of G' -flops starting with $(X/Z, B)$ satisfying

- (1') $\|G_i\| < \epsilon$ where $G_i \geq 0$ is a multiple of G'_i , the birational transform of G' , and
- (2') $(K_{X_i} + B_i + G_i) \cdot R < 0$ for an extremal ray R on X_i ,

we have $(K_{X_i} + B_i) \cdot R \leq 0$. In other words, ϵ is preserved after G' -flops but possibly only in the direction of G' . These claims are proved in [5, Corollary 9, Addendum 4].

Definition 2.3 (Cf., [1, §2]). Let $(X/Z, B)$ be a klt pair, $(Y/Z, B_Y)$ a \mathbb{Q} -factorial klt pair, $\phi: X \dashrightarrow Y/Z$ a birational map such that ϕ^{-1} does not contract divisors, and B_Y the birational transform of B . Moreover, assume that

$$a(D, X, B) \leq a(D, Y, B_Y)$$

for any prime divisor D on birational models of X and assume that the strict inequality holds for any prime divisor D on X which is exceptional/ Y .

We say that $(Y/Z, B_Y)$ is a log minimal model of $(X/Z, B)$ if $K_Y + B_Y$ is nef/ Z . On the other hand, we say that $(Y/Z, B_Y)$ is a Mori fibre space of $(X/Z, B)$ if there is a $K_Y + B_Y$ -negative extremal contraction $Y \rightarrow Y'/Z$ such that $\dim Y' < \dim Y$.

Typically, one obtains a log minimal model or a Mori fibre space by a finite sequence of divisorial contractions and log flips.

Remark 2.4. Let $(X/Z, B)$ be a klt pair and $W \rightarrow X$ a log resolution. Let $B_W = B^\sim + (1 - \epsilon) \sum E_i$ where $0 < \epsilon \ll 1$ and E_i are the exceptional/ X divisors on W . Remember that B^\sim is the birational transform of B . If $(Y/X, B_Y)$ is a log minimal model of $(W/X, B_W)$, which exists by [2], then by the negativity lemma $Y \rightarrow X$ is a small \mathbb{Q} -factorialisation of X . To find a log minimal model or a Mori fibre space of $(X/Z, B)$, it is enough to find one for $(Y/Z, B_Y)$. So, one could assume that X is \mathbb{Q} -factorial by replacing it with Y .

Let $(X/Z, B + C)$ be a \mathbb{Q} -factorial klt pair such that $K_X + B + C$ is nef/ Z . By [1, Lemma 2.6], either $K_X + B$ is nef/ Z or there is an extremal ray R/Z such that $(K_X + B) \cdot R < 0$ and $(K_X + B + \lambda_1 C) \cdot R = 0$ where

$$\lambda_1 := \inf\{t \geq 0 \mid K_X + B + tC \text{ is nef}/Z\}$$

and $K_X + B + \lambda_1 C$ is nef/ Z . Now assume that R defines a divisorial contraction or a log flip $X \dashrightarrow X'$. We can consider $(X'/Z, B' + \lambda_1 C')$ where $B' + \lambda_1 C'$ is the birational transform of $B + \lambda_1 C$ and continue

the argument. That is, either $K_{X'} + B'$ is nef/ Z or there is an extremal ray R'/Z such that $(K_{X'} + B') \cdot R' < 0$ and $(K_{X'} + B' + \lambda_2 C') \cdot R' = 0$ where

$$\lambda_2 := \inf\{t \geq 0 \mid K_{X'} + B' + tC' \text{ is nef}/Z\}$$

and $K_{X'} + B' + \lambda_2 C'$ is nef/ Z . By continuing this process, we obtain a special kind of LMMP on $K_X + B$ which we refer to as the *LMMP with scaling of C* . If it terminates, then we obviously get a log minimal model or a Mori fibre space for $(X/Z, B)$. Note that the required log flips exist by [2].

3. PROOFS

Proof. (of Theorem 1.1) Let $(X/Z, B)$ be a klt pair of dimension d . By Remark 2.4, we can assume that X is \mathbb{Q} -factorial. Let $H \geq 0$ be an \mathbb{R} -divisor which is big/ Z so that $K_X + B + H$ is klt and nef/ Z . Run the LMMP/ Z on $K_X + B$ with scaling of H . If the LMMP terminates, then we get a log minimal model or a Mori fibre space. Suppose that we get an infinite sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips/ Z where we may also assume that $(X_1/Z, B_1) = (X/Z, B)$.

Let λ_i be the threshold on X_i determined by the LMMP with scaling as explained in section 2. So, $K_{X_i} + B_i + \lambda_i H_i$ is nef/ Z , $(K_{X_i} + B_i) \cdot R_i < 0$ and $(K_{X_i} + B_i + \lambda_i H_i) \cdot R_i = 0$ where B_i and H_i are the birational transforms of B and H respectively and R_i is the extremal ray which defines the flipping contraction $X_i \rightarrow Z_i$. Obviously, $\lambda_i \geq \lambda_{i+1}$.

Put $\lambda = \lim_{i \rightarrow \infty} \lambda_i$. If the limit is attained, that is, $\lambda = \lambda_i$ for some i , then the sequence terminates by [2, Corollary 1.4.2]. So, we assume that the limit is not attained. Actually, if $\lambda > 0$, again [2] implies that the sequence terminates. However, we do not need to use [2] in this case. In fact, by replacing B_i with $B_i + \lambda H_i$, we can assume that $\lambda = 0$ hence $\lim_{i \rightarrow \infty} \lambda_i = 0$.

Put $\Lambda_i := B_i + \lambda_i H_i$. Since we are assuming that terminal log flips terminate, or alternatively by [2, Corollary 1.4.3], we can construct a terminal (in codimension ≥ 2) crepant model $(Y_i/Z, \Theta_i)$ of $(X_i/Z, \Lambda_i)$. A slight modification of the argument in Remark 2.4 would do this. Note that we can assume that all the Y_i are isomorphic to Y_1 in codimension one perhaps after truncating the sequence. Let $\Delta_1 = \lim_{i \rightarrow \infty} \Theta_i^\sim$ on Y_1 and let Δ_i be its birational transform on Y_i . The limit is obtained component-wise.

Since H_i is big/ Z and $K_{X_i} + \Lambda_i$ is klt and nef/ Z , $K_{X_i} + \Lambda_i$ and $K_{Y_i} + \Theta_i$ are semi-ample/ Z by the base point freeness theorem for \mathbb{R} -divisors. Thus, $K_{Y_i} + \Delta_i$ is a limit of movable/ Z divisors which in

particular means that it is pseudo-effective/ Z . Note that if $K_{Y_i} + \Delta_i$ is not pseudo-effective/ Z , we get a contradiction by [2, Corollary 1.3.2].

Now run the LMMP/ Z on $K_{Y_1} + \Delta_1$. No divisor will be contracted again because $K_{Y_1} + \Delta_1$ is a limit of movable/ Z divisors. Since $K_{Y_1} + \Delta_1$ is terminal in codimension ≥ 2 , by assumptions, the LMMP terminates with a log minimal model $(W/Z, \Delta)$. By construction, Δ on W is the birational transform of Δ_1 on Y_1 and $G_i := \Theta_i^\sim - \Delta$ on W satisfies $\lim_{i \rightarrow \infty} G_i = 0$.

By Remark 2.2, for each G_i with $i \gg 0$, we can run the LMMP/ Z on $K_W + \Delta + G_i$ which will be a sequence of G_i -flops, that is, $K + \Delta$ would be numerically zero on all the extremal rays contracted in the process. No divisor will be contracted because $K_W + \Delta + G_i$ is movable/ Z . The LMMP ends up with a log minimal model $(W_i/Z, \Omega_i)$. Here, Ω_i is the birational transform of $\Delta + G_i$ and so of Θ_i . Let S_i be the lc model of $(W_i/Z, \Omega_i)$ which is the same as the lc model of $(Y_i/Z, \Theta_i)$ and that of $(X_i/Z, \Lambda_i)$ because $K_{W_i} + \Omega_i$ and $K_{Y_i} + \Theta_i$ are nef/ Z with W_i and Y_i being isomorphic in codimension one, and $K_{Y_i} + \Theta_i$ is the pullback of $K_{X_i} + \Lambda_i$. Also note that since $K_{X_i} + B_i$ is pseudo-effective/ Z , $K_{X_i} + \Lambda_i$ is big/ Z hence S_i is birational to X_i .

By construction $K_{W_i} + \Delta^\sim$ is nef/ Z and it turns out that $K_{W_i} + \Delta^\sim \sim_{\mathbb{R}} 0/S_i$. Suppose that this is not the case. Then, $K_{W_i} + \Delta^\sim$ is not numerically zero/ S_i hence there is some curve C/S_i such that $(K_{W_i} + \Delta^\sim + G_i^\sim) \cdot C = 0$ but $(K_{W_i} + \Delta^\sim) \cdot C > 0$ which implies that $G_i^\sim \cdot C < 0$. Hence, there is a $K_{W_i} + \Delta^\sim + (1 + \tau)G_i^\sim$ -negative extremal ray R/S_i for any $\tau > 0$. This contradicts Remark 2.2 because we must have

$$(K_{W_i} + \Delta^\sim + G_i^\sim) \cdot R = (K_{W_i} + \Delta^\sim) \cdot R = 0$$

Therefore, $K_{W_i} + \Delta^\sim \sim_{\mathbb{R}} 0/S_i$. Now $K_{X_i} + \Lambda_i \sim_{\mathbb{R}} 0/Z_i$ implies that Z_i is over S_i and so $K_{Y_i} + \Delta_i \sim_{\mathbb{R}} 0/S_i$. On the other hand, $K_{X_i} + B_i$ is the pushdown of $K_{Y_i} + \Delta_i$ hence $K_{X_i} + B_i \sim_{\mathbb{R}} 0/S_i$. Thus, $K_{X_i} + B_i \sim_{\mathbb{R}} 0/Z_i$ and this contradicts the fact that $X_i \rightarrow Z_i$ is a $K_{X_i} + B_i$ -flipping contraction. So, the sequence of flips terminates and this completes the proof. \square

Proof. (of Corollary 1.2) Since terminal log flips terminate in dimension 4 by [3][4], the result follows from the Theorem. \square

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